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THE LEADING EDGE EFFECT IN UNSTEADY NATURAL CONVECTION ON A VERTICAL PLATE WITH TIME-DEPENDENT SURFACE TEMPERATURE OR HEAT FLUX

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Abstract—The propagation of the leading edge effect in unsteady natural convection on a semi-infinite vertical plate whose surface was subject to temperature or heat flux increasing exponentially with time or to a power-function change in temperature or in heat flux was analysed by means of the method of Goldstein and Briggs. The influence of the type of transient could be expressed in terms of several time-variables. Approximate expressions applicable also to other types of transient were derived.

NOMENCLATURE

a, thermal diffusivity;
B, dimensionless heat-transfer coefficient defined by equation (42);
B_e, dimensionless heat-transfer coefficient at the end of pure conduction;
B_s, steady dimensionless heat-transfer coefficient;
c, specific heat;
$F(\eta, n)$, function defined by equation (16);
$G(n, Pr)$, function defined by equation (19);
$G(\zeta, Pr)$, function defined by equation (68);
$G^*(\zeta, Pr)$, function, see equation (80);
Gr^*, modified Grashof number;
g, acceleration of gravity;
$H(n, Pr)$, function defined by equation (21);
$H(\zeta, Pr)$, function defined by equation (60);
$H^*(\zeta, Pr)$, function, see equation (81);
h, heat-transfer coefficient;
$I(Pr)$, function, see equation (22);
$J(Pr)$, function, see equation (23);
$K(Pr)$, function, see equation (24);
$N(Pr)$, function, see equation (35);
Nu_x, local Nusselt number;
n, exponent of a power function;
$O(Pr)$, function, see equation (36);
Pr, Prandtl number;
$P(\zeta)$, function, see equation (61);
p, e-folding time;
Q, heat input;
q_w, surface heat flux;
$R(\zeta)$, function, see equation (69);
$S(\zeta, Pr)$, function defined by equation (74);
$S^*(\zeta, Pr)$, function, see equation (82);
$T(\zeta)$, function, see equation (75);
t, elapsed time;
$U(\eta, n)$, dimensionless fluid velocity, see equations (11) and (14);
u, fluid velocity;
$V(\eta, \zeta)$, dimensionless fluid velocity, see equation (56);
$X(\eta, n)$, dimensionless penetration distance, see equations (12) and (13);
x, vertical distance from the leading edge or penetration distance;
x_p, penetration distance;
$x_{p\max}$, maximum penetration distance;
$Y(\eta, \zeta)$, dimensionless penetration distance, see equation (57);
y, horizontal distance from the surface of the plate;
y_0, horizontal distance where the leading edge effect penetrates most deeply;
y_1, horizontal distance where the fluid velocity is maximum.

Greek symbols
α, constant;
β, coefficient of thermal expansion;
γ, constant;
δ, constant;
ε, constant;
ζ, $= \sqrt{(t/p)}$;
η, $= y/2\sqrt{(at)}$;
η_0, $= y_0/2\sqrt{(at)}$;
η_1, $= y_1/2\sqrt{(at)}$;
θ, rise in fluid temperature;
θ_w, surface temperature rise;
κ, constant;
λ, thermal conductivity;
ν, kinematic viscosity;
ρ, density;
$\tau_1, \ldots, \tau_5$, time-variables;
τ_Q, τ_θ, τ_q, τ_{q1}, τ_{q2}, reduced times defined by equations (1) and (47)–(50).

1. INTRODUCTION

UNSTEADY laminar natural convection in the vicinity of a semi-infinite vertical flat plate was first studied by Sugawara and Michiyoshi [1]. Using a method of successive approximations, they solved the boundary-layer equations for a step change in surface temperature. The surrounding fluid was assumed to be initially at rest and to have a uniform temperature. It was found that at the initial stage heat transfer was by purely one-dimensional conduction normal to the surface and that natural convection developed gradually, while the duration necessary for the steady convection to be established was very short.

Siegel [2] utilized the Karman–Pohlhausen type of equations for a step change both in surface temperature and in surface heat flux, and obtained by means of the method of characteristics the time at which one-dimensional conduction terminated and the time at which steady state was reached. He pointed out that the leading edge of the plate was responsible for the transition from conduction to convection. The fluid sufficiently far from the leading edge behaved as if the plate were doubly infinite in length, so that the velocity distribution in this region was independent of the vertical distance and hence the convective heat transfer was zero. The two-dimensional influence which caused the boundary-layer growth to vary with the vertical distance gradually propagated from the leading edge and began to alter the one-dimensional flow configuration at a different time for each position along the plate.

Goldstein and Briggs [3] applied the differential equations for a doubly infinite vertical plate to analyse the propagation of the leading edge effect. This method will be explained in detail later. Nanbu [4] determined the limit of purely one-dimensional conduction on the basis of mathematical singularity which appeared in the boundary-layer equation. His results showed an excellent agreement with those of Goldstein and Briggs.

Goldstein and Eckert [5] experimentally studied the transient process for a step change in heat input. Gebhart and Dring [6] observed the propagation of the leading edge effect and found that it actually travelled up the plate somewhat faster than predicted by Goldstein and Briggs.

All above investigations were made for a step change in surface temperature, surface heat flux and/or heat input. A more general problem is to investigate the influence of the type of transient of enforced quantity, for example, the exponent in the case of a power-function change. Mizukami and Sakurai [7] carried out experiments for two types of transient of heat input, namely, heat input increasing exponentially with time and that increasing linearly with time. The data suggested that the reduced time [8, 9] was important in unsteady natural-convection heat transfer rather than the elapsed time itself since the influence of the type of transient was almost entirely included in this reduced time. The reduced time was defined in terms of the heat input Q and the elapsed time t as follows:

$$\tau_Q = \frac{1}{Q}\int_0^t Q\,\mathrm{d}t_1. \tag{1}$$

In a similar fashion the propagation of the leading edge effect for a transient of surface temperature or of heat flux is also expected to be described by one time-variable. It is this point to be studied in this paper.

In the present analysis, considered are four types of transient, i.e. a power-function change in surface heat flux and in surface temperature, surface temperature increasing exponentially with time and surface heat flux increasing exponentially with time.

2. ANALYTICAL METHOD

The method of Goldstein and Briggs [3] is utilized in the present analysis. As stated in the previous Section, heat transfer at the initial stage is by one-dimensional conduction, and the leading edge is responsible for the transition from conduction to convection. Upon commencement of the unsteady process, therefore, one may consider that the fluid moves up from the leading edge as a wave, in front of which the velocity and the temperature are only functions of the time and the horizontal distance y from the surface of the plate. Behind the wave there must be a dependence on the vertical coordinate x. The basic premises used in the method are that the convective effect will begin at a position x as soon as the fluid which is located initially at the leading edge rises to this position, and that the velocity of this fluid is the same as that of the fluid above it, namely, the velocity predicted from the unsteady-velocity solutions for a doubly infinite vertical plate. The penetration distance x_p of the fluid located initially at the leading edge is a function of t and y. According to the former premise, therefore, the leading edge effect propagates up to the maximum penetration distance $x_{p\max}$.

The governing equations for unsteady laminar natural convection on a doubly infinite vertical flat plate are as follows [3, 10, 11];

$$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2} + g\beta\theta \tag{2}$$

$$\frac{\partial \theta}{\partial t} = a\frac{\partial^2 \theta}{\partial y^2}, \tag{3}$$

where u, θ, ν, β and a are the velocity, the temperature rise, the kinematic viscosity, the coefficient of thermal expansion and the thermal diffusivity of the fluid respectively, and g is the acceleration of gravity. The initial and boundary conditions are

$$\begin{cases} u(y,t) = 0,\ \theta(y,t) = 0 & t \leqq 0 \\ u(\infty,t) = 0,\ u(0,t) = 0,\ \theta(\infty,t) = 0 & t > 0 \end{cases} \tag{4}$$

$$\theta(0,t) = \vartheta_w \quad \text{or} \quad -\lambda\left.\frac{\partial\theta}{\partial y}\right|_{y=0} = q_w \tag{5}$$

where λ, θ_w and q_w are the thermal conductivity of the fluid, the prescribed time-dependent surface temperature rise and heat flux respectively.

Integrating u with respect to t, one obtains the penetration distance x_p. The maximum penetration distance $x_{p\max}$ at any time can be determined by differentiating x_p with respect to y holding t constant and by setting the derivative equal to zero.

According to the above procedure, the penetration distance is given as

$$x_p(y,t) = \int_0^t u(y,t_1)\,\mathrm{d}t_1. \tag{6}$$

Consequently, if y_0 denotes the root of the equation

$$\frac{\partial x_p}{\partial y} = 0, \tag{7}$$

the maximum penetration distance is given as

$$x_{p\max} = x_p(y_0, t). \tag{8}$$

3. A POWER-FUNCTION CHANGE IN SURFACE HEAT FLUX AND IN SURFACE TEMPERATURE

It is noted that when the transient of surface heat flux is prescribed as

$$q_w \propto t^n, \tag{9}$$

the surface temperature rise also varies as a power function of time [12] such that

$$\theta_w = \frac{\Gamma(n+1)}{\sqrt{(\rho c\lambda)}\Gamma(n+3/2)} q_w \sqrt{t} \propto t^{n+1/2}, \tag{10}$$

where ρ is the density of the fluid, and $\Gamma(n)$ is Gamma function.

If $2n$ is an integer greater than -2, the velocity and the penetration distance are expressed as follows;

$$\frac{\sqrt{(\rho c\lambda)}u}{g\beta q_w t^{3/2}} = U(\eta, n) \tag{11}$$

$$\frac{\sqrt{(\rho c\lambda)}x_p}{g\beta q_w t^{5/2}} = X(\eta, n), \tag{12}$$

where for Prandtl number Pr not equal to unity

$$X(\eta, n) = \frac{2^{2n+5}\Gamma(n+1)}{1-Pr} \times \left\{ i^{2n+5}\operatorname{erfc}(\eta) - i^{2n+5}\operatorname{erfc}\left(\frac{\eta}{\sqrt{Pr}}\right)\right\} \tag{13a}$$

and for Prandtl number equal to unity

$$X(\eta, n) = 2^{2n+4}\Gamma(n+1)\eta i^{2n+4}\operatorname{erfc}(\eta) \tag{13b}$$

and

$$U(\eta, n) = \frac{\Gamma(n+1)}{\Gamma(n)} X(\eta, n-1) \tag{14}$$

with

$$\eta = \frac{y}{2\sqrt{(at)}}$$

$$i^k\operatorname{erfc}(\eta) = \int_\eta^\infty i^{k-1}\operatorname{erfc}(z)\,dz \quad (k = 1, 2, 3, \ldots)$$

$$i^0\operatorname{erfc}(\eta) = \operatorname{erfc}(\eta) = \frac{2}{\sqrt{\pi}}\int_\eta^\infty \exp(-z^2)\,dz.$$

Equation (7) is written in terms of η as

$$\frac{\partial x_p}{\partial y} \propto \frac{\partial X}{\partial \eta} \propto F(\eta, n) = 0, \tag{15}$$

where

$$F(\eta, n) = \begin{cases} \dfrac{1}{\sqrt{Pr}} i^{2n+4}\operatorname{erfc}\left(\dfrac{\eta}{\sqrt{Pr}}\right) - i^{2n+4}\operatorname{erfc}(\eta), & Pr \neq 1 \\ i^{2n+4}\operatorname{erfc}(\eta) - \eta i^{2n+3}\operatorname{erfc}(\eta), & Pr = 1. \end{cases} \tag{16}$$

Substituting the root η_0 of equation (15) for η in equation (12), one obtains the relationship between the elapsed time and the maximum penetration distance as follows;

$$\frac{\sqrt{(\rho c\lambda)}x_{p\max}}{g\beta q_w t^{5/2}} = X(\eta_0, n). \tag{17}$$

By replacing $x_{p\max}$ in equation (17) by x and by rearranging it, the relationship between the time t and the position x up to which the leading edge effect propagates can be written as

$$\left\{\frac{g\beta q_w}{\sqrt{(\rho c\lambda)}x}\right\}^{2/5} t = G(n, Pr), \tag{18}$$

where

$$G(n, Pr) = \{X(\eta_0, n)\}^{-2/5}. \tag{19}$$

By virtue of equation (10), equation (18) can be expressed in an alternative form as

$$\left(\frac{g\beta\theta_w}{x}\right)^{1/2} t = H(n, Pr), \tag{20}$$

where

$$H(n, Pr) = \left\{\frac{\Gamma(n+3/2)}{\Gamma(n+1)} X(\eta_0, n)\right\}^{-1/2} \tag{21}$$

Numerical calculation of the values of η_0, $G(n, Pr)$, $H(n, Pr)$ was performed for a number of combinations of Pr and n on an electronic digital computer. Some of the results are shown in Figs. 1–3. These figures suggest that the following good approximate expressions hold:

$$G(n, Pr) = (n+\alpha)I(Pr) \tag{22}$$

$$H(n, Pr) = (n+\gamma)J(Pr) \tag{23}$$

$$\frac{1}{\eta_0^2} = (n+\delta)K(Pr), \tag{24}$$

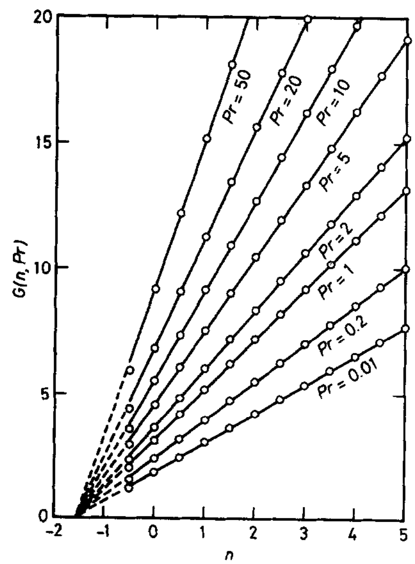

FIG. 1. Dependence of $G(n, Pr)$ on n.

Table 1. Values of the constants and the functions

Pr	$I(Pr)$	α	$J(Pr)$	γ	$K(Pr)$	δ	$N(Pr)$	ε	$O(Pr)$	κ
0.01	1.17	1.58	1.20	1.90	60.7	3.02	1.27	1.17	1.42	1.44
0.02	1.22	1.58	1.26	1.90	38.3	2.96	1.36	1.17	1.57	1.43
0.05	1.31	1.57	1.38	1.89	21.4	2.90	1.53	1.16	1.88	1.42
0.1	1.41	1.57	1.51	1.89	14.0	2.86	1.72	1.16	2.23	1.42
0.2	1.53	1.57	1.68	1.89	9.39	2.83	1.98	1.15	2.76	1.41
0.5	1.76	1.57	2.00	1.89	5.69	2.80	2.50	1.15	3.93	1.40
1	2.00	1.57	2.35	1.89	3.98	2.80	3.10	1.15	5.42	1.40
2	2.32	1.57	2.83	1.89	2.84	2.80	3.97	1.15	7.85	1.40
5	2.91	1.57	3.75	1.89	1.88	2.83	5.79	1.15	13.8	1.41
10	3.53	1.57	4.77	1.89	1.40	2.86	7.98	1.16	22.3	1.42
20	4.34	1.57	6.18	1.89	1.07	2.90	11.3	1.16	37.5	1.42
50	5.84	1.58	8.94	1.90	0.766	2.96	18.5	1.17	78.5	1.43
100	7.39	1.58	12.0	1.90	0.607	3.02	27.4	1.17	142	1.44

where α, γ and δ are constants, and $I(Pr)$, $J(Pr)$ and $K(Pr)$ are functions of Pr. Values of the constants and functions were determined for each value of Pr by means of the least square technique, and are tabulated in Table 1. From this table one obtains

$$\alpha = 1.57 \tag{25}$$

$$\gamma = 1.89 \tag{26}$$

$$\delta = 2.9. \tag{27}$$

By introducing variables τ_1, τ_2 and τ_3 defined as

$$\tau_1 = \frac{t}{n+\alpha} \tag{28}$$

$$\tau_2 = \frac{t}{n+\gamma} \tag{29}$$

$$\tau_3 = \frac{t}{n+\delta}, \tag{30}$$

equations (18), (20) and (24) become

$$\left\{\frac{g\beta q_w}{\sqrt{(\rho c\lambda)x}}\right\}^{2/5}\tau_1 = I(Pr) \tag{31}$$

$$\left(\frac{g\beta\theta_w}{x}\right)^{1/2}\tau_2 = J(Pr) \tag{32}$$

$$\frac{4a\tau_3}{y_0^2} = K(Pr). \tag{33}$$

These equations express the influence of the type of transient of surface temperature and of surface heat flux on the propagation of the leading edge effect in a simple manner through variables τ_1, τ_2 and τ_3 only.

If $\eta_1[=y_1/\sqrt{(at)}]$ is the root of the equation

$$\frac{\partial u}{\partial y} \propto \frac{\partial U}{\partial \eta} \propto F(\eta, n-1) = 0, \tag{34}$$

the velocity takes the maximum value u_{max} at $\eta = \eta_1$. After similar rearrangement and calculation, it was found that the following approximate expressions hold:

$$\left\{\frac{g\beta q_w}{\sqrt{(\rho c\lambda)}u_{max}}\right\}^{2/3}\tau_4 = N(Pr) \tag{35}$$

$$\frac{g\beta\theta_w}{u_{max}}\tau_5 = O(Pr), \tag{36}$$

where $N(Pr)$ and $O(Pr)$ are functions of Pr, and

$$\tau_4 = \frac{t}{n+\varepsilon} \tag{37}$$

$$\tau_5 = \frac{t}{n+\kappa} \tag{38}$$

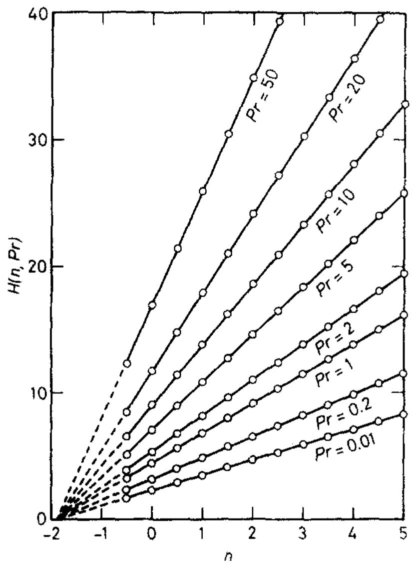

FIG. 2. Dependence of $H(n, Pr)$ on n.

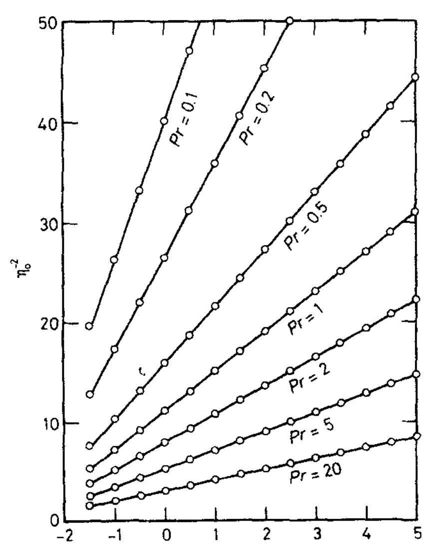

FIG. 3. Dependence of η_0 on n.

with ε and κ as constants. Values of $N(Pr)$, $O(Pr)$, ε and κ for each value of Pr are shown in Table 1. From this table

$$\varepsilon = 1.16 \tag{39}$$

$$\kappa = 1.42. \tag{40}$$

Thus the influence of the type of transient on the maximum velocity again is almost entirely included in τ_4 and τ_5. Referring to equations (15), (24) and (34), one readily obtains the following equation from which the root η_1 can approximately be evaluated:

$$\frac{1}{\eta_1^2} = (n-1+\delta)K(Pr). \tag{41}$$

The propagation of the leading edge effect and the related quantities were found to be simply formulated by introducing the time-variables τ_1 through τ_5, which almost entirely include the influence of n, namely, the influence of the type of transient of surface heat flux and temperature.

Now consider the dimensionless heat-transfer coefficient

$$B = \frac{Nu_x}{(Gr^*)^{1/5}}. \tag{42}$$

Nu_x and Gr^* are the local Nusselt number and the modified Grashof number respectively, and are defined as

$$Nu_x = \frac{hx}{\lambda} \tag{43}$$

$$Gr^* = \frac{g\beta q_w x^4}{\lambda v^2}, \tag{44}$$

where h is the heat-transfer coefficient. If B_e and B_s denote the dimensionless heat-transfer coefficient at the end of pure conduction and the steady one corresponding to the same surface heat flux respectively, then the difference between B_s and B_e for a step change in the surface heat flux is known as "overshoot". The heat-transfer coefficient at the end of pure conduction is obtained from equation (10) as

$$h = \frac{q_w}{\theta_w} = \frac{\Gamma(n+3/2)\sqrt{(\rho c\lambda)}}{\Gamma(n+1)\sqrt{t}}. \tag{45}$$

Elimination of t from equations (18) and (45) and rearrangement yield

$$B_e = \frac{\Gamma(n+3/2)Pr^{2/5}}{\Gamma(n+1)\sqrt{G(n, Pr)}}. \tag{46}$$

Table 2. A comparison between the dimensionless heat-transfer coefficient at the end of pure conduction and the steady one corresponding to the same surface heat flux

Pr	B_s	B_e			
		$n = 0$	$n = 0.5$	$n = 1$	$n = 2$
0.1	0.263	0.238	0.263	0.277	0.294
1	0.534	0.501	0.553	0.584	0.620
10	0.944	0.948	1.045	1.104	1.172
100	1.556	1.641	1.811	1.914	2.032

Values of B_s were given in [13] for several values of Pr. A comparison between B_e and B_s is made in Table 2. As n in increased, B_e tends to increase. For a step change and for smaller values of Pr, B_e is less than B_s.

4. APPROXIMATE EXPRESSIONS

Since the surface heat flux is a power function of time, the following rough approximate expressions hold for the variables τ_1 through τ_5 introduced in the previous section:

$$\tau_1 \simeq \tau_\theta = \frac{1}{\theta_w}\int_0^t \theta_w \,dt_1 \tag{47}$$

$$\tau_2 \simeq \tau_{q1} = \frac{\int_0^t \int_0^{t_1} q_w \,dt_2 \,dt_1}{\int_0^t q_w \,dt_1} \tag{48}$$

$$\tau_3 \simeq \tau_{q2} = \frac{\int_0^t \int_0^{t_1} \int_0^{t_2} q_w \,dt_3 \,dt_2 \,dt_1}{\int_0^t \int_0^{t_1} q_w \,dt_2 \,dt_1} \tag{49}$$

$$\tau_4 \simeq \tau_q = \frac{1}{q_w}\int_0^t q_w \,dt_1 \tag{50}$$

$$\tau_5 \simeq \tau_\theta. \tag{51}$$

The reduced times τ_θ, τ_q, τ_{q1} and τ_{q2}, which are very similar to the reduced time defined by equation (1), depend only on the surface heat flux or temperature. By combining, for example, equations (47), (48) and (49) with equations (31), (32) and (33) respectively, useful approximate equations which describe the propagation of the leading edge effect can be obtained as follows;

$$\left\{\frac{g\beta q_w}{\sqrt{(\rho c\lambda)}x}\right\}^{2/5} \tau_\theta = I(Pr) \tag{52}$$

$$\left(\frac{g\beta\theta_w}{x}\right)^{1/2} \tau_{q1} = J(Pr) \tag{53}$$

$$\frac{4a\tau_{q2}}{y_0^2} = K(Pr). \tag{54}$$

The above equations, however, have a somewhat inconvenient aspect such that in utilizing, for example, equation (52) τ_θ must be computed from the change in surface temperature, although the surface heat flux is prescribed.

5. SURFACE TEMPERATURE INCREASING EXPONENTIALLY WITH TIME

The reduced times in the previous section seem to play a significant role also in the unsteady natural convection induced by other types of transient of surface temperature or of heat flux. The approximate expressions derived in the previous section are of the form applicable in many other cases, which is illustrated in this chapter for the case of an exponentially increasing surface temperature, i.e. $\theta_w \propto \exp(t/p)$, where p is the e-folding time. It is noted that in this case

$$q_w = \frac{\sqrt{(\rho c\lambda)}}{\sqrt{p}}\left\{\text{erf}(\zeta) + \frac{1}{\sqrt{\pi}\zeta}\exp(-\zeta^2)\right\}\theta_w, \tag{55}$$

where

$$\zeta = \sqrt{(t/p)}$$

$$\operatorname{erf}(\zeta) = \frac{2}{\sqrt{\pi}} \int_0^{\zeta} \exp(-z^2)\,dz.$$

The reduced times are zero when ζ is zero, and tend to p as ζ becomes large.

The velocity and the penetration distance are obtained as

$$\frac{u}{g\beta p\theta_w} = V(\eta, \zeta) \tag{56}$$

$$\frac{x_p}{g\beta p^2\theta_w} = Y(\eta, \zeta), \tag{57}$$

where $V(\eta, \zeta)$ and $Y(\eta, \zeta)$ are functions of η and ζ, but are too lengthy to be written here. Substitution of the root η_0 of the equation

$$\frac{\partial x_p}{\partial y} \propto \frac{\partial}{\partial \eta} Y(\eta, \zeta) = 0 \tag{58}$$

for η in equation (52) yields an expression similar to equation (20):

$$\left(\frac{g\beta\theta_w}{x}\right)^{1/2} p = H(\zeta, Pr), \tag{59}$$

where

$$H(\zeta, Pr) = \{Y(\eta_0, Pr)\}^{-1/2}. \tag{60}$$

Values of η_0 and $H(\zeta, Pr)$ were computed for a number of combinations of ζ and Pr. The reciprocal of $H(\zeta, Pr)$ increased and tended to a certain finite value as ζ became large. The ratio $H(\zeta = \infty, Pr)/H(\zeta, Pr)$ is shown in Table 3 together with τ_{q1}/p. In practice, $H(\zeta = 5, Pr)$ was used instead of $H(\zeta = \infty, Pr)$. This table indicates that the ratio is almost independent of Pr, that is, it can be written as

$$\frac{H(\zeta = \infty, Pr)}{H(\zeta, Pr)} = P(\zeta), \tag{61}$$

where $P(\zeta)$ is a function of ζ only. Now let τ_2 be defined as follows;

$$\tau_2 = pP(\zeta). \tag{62}$$

Then, eliminating $H(\zeta, Pr)$ from equations (59) and (61) and using equation (62), one obtains

$$\left(\frac{g\beta\theta_w}{x}\right)^{1/2} \tau_2 = H(\zeta = \infty, Pr). \tag{63}$$

This equation, as well as equation (32), indicates that the influence of the type of transient is entirely included in one time-variable τ_2. Table 6 shows the values of $H(\zeta = \infty, Pr)$ together with the values of $J(Pr)$ obtained in Section 3. As readily seen from this table, the following equation holds within 1%:

$$H(\zeta = \infty, Pr) = J(Pr). \tag{64}$$

Table 3 implies also that

$$P(\zeta) \simeq \frac{\tau_{q1}}{p} \tag{65}$$

or

$$\tau_2 \simeq \tau_{q1}. \tag{66}$$

Therefore the approximate expression, equation (53), is valid also for surface temperature increasing exponentially with time.

Elimination of θ_w from equations (63) and (55) yields

$$\left\{\frac{g\beta q_w}{\sqrt{(\rho c\lambda)x}}\right\}^{2/5} p = G(\zeta, Pr), \tag{67}$$

where

$$G(\zeta, Pr) = \{H(\zeta, Pr)\}^{4/5} \left\{\operatorname{erf}(\zeta) + \frac{\exp(-\zeta^2)}{\sqrt{\pi}\zeta}\right\}^{2/5}. \tag{68}$$

The ratio $G(\zeta = \infty, Pr)/G(\zeta, Pr)$ is demonstrated in Table 4 together with τ_θ/p. Again one may set

$$\frac{G(\zeta = \infty, Pr)}{G(\zeta, Pr)} = R(\zeta), \tag{69}$$

where $R(\zeta)$ is a function of ζ only, which is nearly equal to τ_θ/p. From equations (67) and (69)

$$\left\{\frac{g\beta q_w}{\sqrt{(\rho c\lambda)x}}\right\}^{2/5} \tau_1 = G(\zeta = \infty, Pr), \tag{70}$$

where

$$\tau_1 = pR(\zeta) \simeq \tau_\theta. \tag{71}$$

Further, as seen from Table 6,

$$G(\zeta = \infty, Pr) = I(Pr). \tag{72}$$

Therefore, equation (31) is applicable if the variable τ_1 is defined by equation (71). Equation (52) is valid without any modification. It is seen from equation (68) that

$$G(\zeta = \infty, Pr) = \{H(\zeta = \infty, Pr)\}^{4/5}. \tag{73}$$

Define a function $S(\zeta, Pr)$ as

$$S(\zeta, Pr) = \frac{1}{\zeta^2\eta_0^2}. \tag{74}$$

Then $S(\zeta, Pr)$ has features similar to those of $H(\zeta, Pr)$ and $G(\zeta, Pr)$. The ratio $S(\zeta = \infty, Pr)/S(\zeta, Pr)$ is tabulated in Table 5 together with τ_{q2}/p. It can be written as

$$\frac{S(\zeta = \infty, Pr)}{S(\zeta, Pr)} = T(\zeta) \simeq \frac{\tau_{q2}}{p}. \tag{75}$$

If the variable τ_3 is defined as

$$\tau_3 = pT(\zeta) \simeq \tau_{q2}, \tag{76}$$

combination of equations (74)–(76) yeilds

$$\frac{4a\tau_3}{y_0^2} = S(\zeta = \infty, Pr). \tag{77}$$

Further, as seen from Table 6,

$$S(\zeta = \infty, Pr) = K(Pr). \tag{78}$$

Therefore equations (33) and (54) are also applicable.

Table 3. Values of $H(\zeta = \infty, Pr)/H(\zeta, Pr)$ for $\theta_w \propto \exp(\zeta^2)$ and comparison with τ_{q1}/p

ζ	$Pr = 0.001$	$Pr = 0.01$	$Pr = 0.1$	$Pr = 1$	$Pr = 10$	$Pr = 100$	$Pr = 1000$	τ_{q1}/p
0.25	0.044	0.044	0.044	0.044	0.044	0.044	0.044	0.041
0.5	0.164	0.165	0.167	0.167	0.167	0.165	0.164	0.156
0.75	0.334	0.336	0.338	0.339	0.338	0.336	0.334	0.332
1	0.518	0.521	0.524	0.525	0.524	0.521	0.518	0.507
1.25	0.685	0.688	0.692	0.693	0.691	0.688	0.685	0.680
1.5	0.815	0.818	0.821	0.822	0.821	0.818	0.815	0.815
1.75	0.904	0.906	0.908	0.909	0.908	0.906	0.904	0.906
2	0.956	0.957	0.958	0.959	0.958	0.957	0.956	0.958
2.25	0.982	0.983	0.983	0.984	0.983	0.983	0.982	0.984
2.5	0.994	0.994	0.994	0.994	0.994	0.994	0.994	0.995

Table 4. Values of $G(\zeta = \infty, Pr)/G(\zeta, Pr)$ for $\theta_w \propto \exp(\zeta^2)$ and comparison with τ_θ/p

ζ	$Pr = 0.001$	$Pr = 0.01$	$Pr = 0.1$	$Pr = 1$	$Pr = 10$	$Pr = 100$	$Pr = 1000$	τ_θ/p
0.25	0.058	0.058	0.058	0.058	0.058	0.058	0.058	0.061
0.5	0.206	0.207	0.208	0.209	0.208	0.207	0.206	0.221
0.75	0.395	0.397	0.399	0.400	0.399	0.397	0.395	0.430
1	0.579	0.582	0.585	0.586	0.585	0.582	0.579	0.632
1.25	0.734	0.736	0.739	0.740	0.739	0.736	0.734	0.790
1.5	0.847	0.849	0.852	0.853	0.852	0.849	0.847	0.895
1.75	0.922	0.923	0.925	0.926	0.925	0.923	0.922	0.953
2	0.964	0.965	0.966	0.967	0.966	0.965	0.964	0.982
2.25	0.986	0.986	0.987	0.987	0.987	0.986	0.986	0.994
2.5	0.995	0.995	0.995	0.995	0.995	0.995	0.995	0.998

Table 5. Values of $S(\zeta = \infty, Pr)/S(\zeta, Pr)$ for $\theta_w \propto \exp(\zeta^2)$ and comparison with τ_{q2}/p

ζ	$Pr = 0.001$	$Pr = 0.01$	$Pr = 0.1$	$Pr = 1$	$Pr = 10$	$Pr = 100$	$Pr = 1000$	τ_{q2}/p
0.25	0.023	0.025	0.026	0.027	0.026	0.025	0.023	0.025
0.5	0.089	0.097	0.103	0.105	0.103	0.097	0.089	0.097
0.75	0.193	0.210	0.223	0.228	0.223	0.210	0.193	0.210
1	0.328	0.353	0.373	0.380	0.373	0.353	0.328	0.353
1.25	0.480	0.512	0.535	0.544	0.535	0.512	0.480	0.509
1.5	0.633	0.666	0.690	0.699	0.690	0.666	0.633	0.660
1.75	0.768	0.798	0.818	0.825	0.818	0.798	0.768	0.789
2	0.872	0.893	0.907	0.912	0.907	0.893	0.872	0.884
2.25	0.939	0.952	0.959	0.962	0.959	0.952	0.939	0.945
2.5	0.975	0.981	0.985	0.986	0.985	0.981	0.976	0.977

Table 6. Comparisons of $H(\zeta = \infty, Pr)$ with $J(Pr)$, of $G(\zeta = \infty, Pr)$ with $I(Pr)$, and of $S(\zeta = \infty, Pr)$ with $K(Pr)$

Pr	$H(\zeta = \infty, Pr)$	$J(Pr)$	$G(\zeta = \infty, Pr)$	$I(Pr)$	$S(\zeta = \infty, Pr)$	$K(Pr)$
0.001	1.07	1.08	1.06	1.08	314	312
0.01	1.19	1.20	1.15	1.17	61.1	60.7
0.1	1.50	1.51	1.38	1.41	14.1	14.0
1	2.33	2.35	1.97	2.00	4.00	3.98
10	4.73	4.77	3.47	3.53	1.41	1.40
100	11.9	12.0	7.26	7.39	0.611	0.607
1000	34.0	34.2	16.8	17.1	0.314	0.312

6. SURFACE HEAT FLUX INCREASING EXPONENTIALLY WITH TIME

If the surface heat flux is increased exponentially with time, i.e. $q_w \propto \exp(\zeta^2)$, then the surface temperature rise is related to the surface heat flux by the following equation;

$$\theta_w = \frac{\sqrt{p}}{\sqrt{(\rho c \lambda)}} q_w \operatorname{erf}(\zeta). \tag{79}$$

The propagation of the leading edge effect is again described as

$$\left\{\frac{g\beta q_w}{\sqrt{(\rho c\lambda)}x}\right\}^{2/5} p = G^*(\zeta, Pr) \tag{80}$$

$$\left(\frac{g\beta\theta_w}{x}\right)^{1/2} p = H^*(\zeta, Pr) \tag{81}$$

$$\frac{1}{\zeta^2\eta_0^2} = S^*(\zeta, Pr), \tag{82}$$

where $G^*(\zeta, Pr)$, $H^*(\zeta, Pr)$ and $S^*(\zeta, Pr)$ are functions of ζ and Pr. As a result of calculation, the behaviors of $G^*(\zeta, Pr)$, $H^*(\zeta, Pr)$ and $S^*(\zeta, Pr)$ were found quite similar to those of $G(\zeta, Pr)$, $H(\zeta, Pr)$ and $S(\zeta, Pr)$ respectively. Consequently, the same type of equations and conclusions as those derived in Section 5 were obtained. In particular, the values of $G^*(\zeta = \infty, Pr)$, $H^*(\zeta = \infty, Pr)$ and $S^*(\zeta = \infty, Pr)$ precisely accorded with those of $G(\zeta = \infty, Pr)$, $H(\zeta = \infty, Pr)$ and $S(\zeta = \infty, Pr)$ respectively. This result is expected since the surface temperature rise is proportional to the surface heat flux when ζ is sufficiently large.

7. SUMMARY OF THE RESULTS

The propagation of the leading edge effect in unsteady natural convection on a semi-infinite vertical plate was analysed for a power-function change and also for an exponential-function change in surface heat flux and in surface temperature. It was found that the influence of the type of transient can be expressed in terms of time-variables, each of which is nearly equal to one of the reduced times. The utility of approximate expressions in terms of the reduced times was illustrated.

8. CONCLUDING REMARKS

In this analysis the leading edge effect is assumed to propagate with the velocity determined from the temperature profile induced by unsteady conduction. Therefore the transition of the heat-transfer regime from conduction to convection is closely related with the fashion how this profile varies with time. This is also the case as to the stability of fluid layer with a time-dependent temperature profile. The onset of convection in fluid unsteadily heated from below was found to depend on the reduced time τ_0 [14]. This arises because there exists, in an approximate sense, one-to-one correspondence between a temperature profile and a value of the reduced time in the unsteady conduction heat transfer [15].

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EFFET DE BORD D'ATTAQUE DANS UNE CONVECTION NATURELLE INSTATIONNAIRE SUR UNE PLAQUE VERTICALE, AVEC TEMPERATURE DE SURFACE OU FLUX THERMIQUE DEPENDANT DU TEMPS

Résumé—Par la méthode de Goldstein et Briggs, on analyse la propagation de l'effet de bord d'attaque dans la convection naturelle sur une plaque verticale et semi-infinie dont la surface est soumise à une température ou à un flux de chaleur croissant exponentiellement avec le temps, ou bien à un changement en fonction-puissance de la température ou du flux. L'influence du type d'évolution peut être exprimée en termes de plusieurs variables temporelles. On donne des expressions approchées applicables à d'autres types d'évolution.

DER ANSTRÖMKANTENEFFEKT BEI INSTATIONÄRER, NATÜRLICHER KONVEKTION AN EINER VERTIKALEN PLATTE MIT ZEITABHÄNGIGER OBERFLÄCHENTEMPERATUR ODER WÄRMESTROMDICHTE

Zusammenfassung—Nach den Methoden von Goldstein und Briggs wird die Fortpflanzung des Anströmkanteneffekts bei instationärer, natürlicher Konvektion an einer halbunendlichen, vertikalen Platte untersucht. Für die Oberflächentemperatur oder die Wärmestromdichte wurde eine zeitliche Zunahme nach einer Exponential—bzw. Potenzfunktion angesetzt. Der Einfluß der Art der Transiente konnte durch verschiedene Zeitvariablen erfaßt werden. Für andere Transienten wurden Näherungslösungen abgeleitet.

ВЛИЯНИЕ ПЕРЕДНЕЙ КРОМКИ НА ПРОЦЕСС НЕСТАЦИОНАРНОЙ ЕСТЕСТВЕННОЙ КОНВЕКЦИИ НА ВЕРТИКАЛЬНОЙ ПЛАСТИНЕ ПРИ ИЗМЕНЯЕМОЙ СО ВРЕМЕНЕМ ТЕМПЕРАТУРЕ ПОВЕРХНОСТИ ИЛИ ВЕЛИЧИНЕ ТЕПЛОВОГО ПОТОКА

Аннотация — С помощью метода Гольдштейна и Бриггса анализируется эффект передней кромки при нестационарной естественной конвекции на полубесконечной вертикальной пластине, температура поверхности которой или величина подводимого теплового потока увеличиваются экспоненциально со временем или изменяются по степенной функции. Влияние типа нестационарности можно описать с помощью нескольких временных переменных. Приближенные выражения применимы также для других нестационарных условий.